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Selective withdrawal from a layered fluid

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The selective withdrawal of a layered fluid from a reservoir has interesting properties which are not present if the stratification is continuous. The problem investigated is that of finding the ratios of the discharges from each flowing layer when the total discharge from all layers is regulated at the outlet. It is shown that the ratios are determined by the requirement that the flow be smooth at critical points in the flow, each critical point being a point at which the long-wave velocity on one of the interfaces is zero. If there are too many critical points, the flow is over-determined and becomes unsteady. It follows therefore that if the external conditions change slowly, such as in the slow draining of a reservoir, the flow alternates between intervals in which the flow ratios also change slowly and intervals in which the flow ratios oscillate unsteadily. An experiment is described in which the theoretical conclusions were tested for two-layer flow. Good agreement was obtained between theory and experiment.

1. Introduction

It is of considerable practical importance to understand all aspects of the selective withdrawal of a stably stratified fluid from a reservoir. Many of the problems involved are discussed in the review by Brooks & Koh (1969). The properties of the fluid discharged are determined by the properties of the fluid in the reservoir, by the geometry of the reservoir and by the total discharge through the outlet.

Wood (1968) analysed the flow from a stably stratified reservoir through a horizontal contraction into an open channel. He showed that two-layer flow is determined by the geometry of the section of minimum width in the contraction together with the geometry of a section upstream of this point. It is well known that one-layer flow from a reservoir is determined solely by the geometry of the section of minimum width, which is referred to as the point of control. Wood showed that, for two-layer flow, there exists a second point of control, named the point of virtual control. He carried out experiments which confirmed the theoretical predictions.

Similar methods of analysis have been applied to lock exchange flows (Wood

1970), layered fluid flow over a weir (Wood & Lai 1972*a*) and layered fluid flow from a reservoir into a closed conduit (Wood & Lai 1972*b*). The present investigation is concerned with setting up a theory for multi-layered flow from a reservoir into a closed conduit, and with testing experimentally the theoretical conclusions.

The common assumptions in all these models are that the effect of viscous forces on the flow properties is negligible, that the velocity profiles in each layer remain nearly uniform (particularly in the neighbourhood of points of control), and that the flow is steady. There is some modification of the uniform flow by viscous forces at solid boundaries and between layers, but attention is confined to situations where this effect is negligible. The steady-flow assumption means that the time taken for a particle to travel from the reservoir through the conduit is small compared with the time taken for streamline patterns to change as the reservoir drains.

2. Theory

The model analysed here is that of a stably stratified n-layer fluid in a reservoir being withdrawn through a contraction with a closed conduit. The notation for a two-layer flow is sketched in figure 1. The base of the model is the horizontal plane $y = 0$, x is the horizontal variable in the flow direction, and the height and breadth of the contraction are $y = Y(x)$ and $z = B(x)$, where $Y'(x)$ and $B'(x)$ are everywhere negative. The layer densities in order of increasing height and decreasing magnitude are $\rho_1, \rho, \ldots, \rho_n$, the corresponding reservoir depths are $h_1, h_2, \ldots, h_n$, and the layer depths are $y_1(x), y_2(x), \ldots, y_n(x)$. The flow is assumed to be steady with volume fluxes $Q_1, Q_2, \ldots, Q_n$ and a total volume flux Q. When k layers of fluid are flowing through the contraction, the interface between the flowing kth layer and the stationary $(k+1)$th layer ends on contact with the roof of the contraction. In figure 1, for example, where there are two flowing layers, this point of contact is denoted by C_2. When there are k flowing layers, the corresponding point of contact is denoted by C_k.

Wood & Lai (1972*b*) investigated theoretically and confirmed experimentally the model for single-layer flow and the transition to two-layer flow. They determined the maximum single-layer flow and showed that it occurs when the interface is tangential to the roof of the contraction at the contact point C_1, that is $dy_1/dx = dY/dx$ there. The energy equation for single-layer flow upstream of C_1 is

$$\tfrac{1}{2}\rho_1 Q_1^2/(B^2 y_1^2) = (\rho_1-\rho_2)\, g(h_1-y_1). \tag{2.1}$$

This equation holds at the point of contact, where $y_1 = Y$, and for given upstream conditions may be differentiated with respect to Y to determine the maximum Q_1. It then yields

$$\frac{Y}{h_1} = \left(1+\frac{Y}{B}\frac{dB}{dY}\right)\bigg/\left(\frac{3}{2}+\frac{Y}{B}\frac{dB}{dY}\right) \tag{2.2}$$

at C_1. Considerable simplification is achieved here and in subsequent calculations if attention is confined to contractions whose shape satisfies $YdB/BdY =$ constant. For this reason, the present theory and experiments relate to contractions

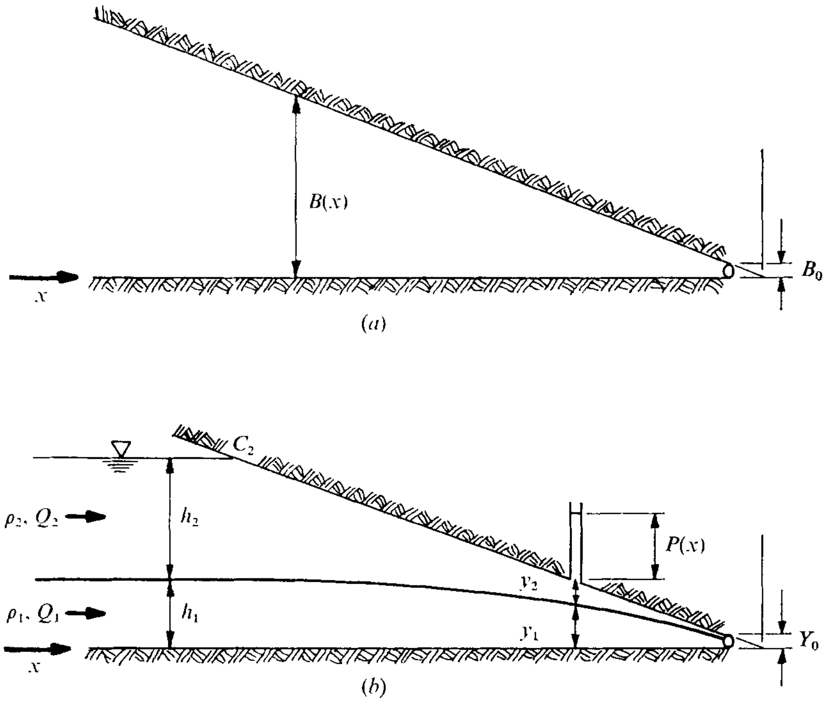

FIGURE 1. Notation. (*a*) Plan. (*b*) Elevation.

of square cross-section, for which (2.2) becomes $Y = \frac{4}{5}h_1$ at the contact point C_1. The maximum single-layer volume flux then satisfies

$$\frac{\rho_1^{\frac{1}{2}}Q}{(\rho_1-\rho_2)^{\frac{1}{2}}g^{\frac{1}{2}}h_1^{\frac{5}{2}}} = \left(\frac{4}{5}\right)^2\left(\frac{2}{5}\right)^{\frac{1}{2}}. \tag{2.3}$$

When two layers are flowing, the equation for the energy difference across the interface inside the contraction downstream of the contact point C_2 is

$$\tfrac{1}{2}\rho_1 Q_1^2/(B^2y_1^2) - \tfrac{1}{2}\rho_2 Q_2^2/(B^2y_2^2) = (\rho_1-\rho_2)g(h_1-y_1). \tag{2.4}$$

This equation, together with the total-depth equation

$$y_1+y_2 = Y(x) \quad (= B(x)), \tag{2.5}$$

determines $y_1(x)$ and $y_2(x)$ downstream of C_2 once Q_1 and Q_2 are known. The condition for smooth flow downstream of C_2 is that the derivatives y_{1x} and y_{2x} exist at all points in this region. The derivatives satisfy

$$\begin{pmatrix} (\rho_1-\rho_2)g-\rho_1 Q_1^2/(B^2y_1^3) & \rho_2 Q_2^2/(B^2y_2^3) \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_{1x} \\ y_{2x} \end{pmatrix} = \begin{pmatrix} \rho_1 Q_1^2/B^3y_1^2) - \rho_2 Q_2^2/(B^3y_2^2) \\ dY/dB \end{pmatrix} B_x. \tag{2.6}$$

The 2×2 matrix is singular at a critical point, this being a point at which the velocity of long waves on the interface is zero, or equivalently, at which the internal Froude number is unity.† The condition that the derivatives exist at a critical point is that the coefficients in the two rows of (2.6) be proportional, that is

$$\begin{aligned}(\rho_1-\rho_2)\,g-\rho_1 Q_1^2/(B^2y_1^3) &= \rho_2 Q_2^2/(B^2y_2^3) \\ &= (\rho_1 Q_1^2/(B^3y_1^2)-\rho_2 Q_2^2/(B^3y_2^2))\,dB/dY.\end{aligned} \tag{2.7}$$

Equations (2.4), (2.5) and (2.7), together with the equation for the total volume flux

$$Q_1+Q_2 = Q \quad \text{(given)}, \tag{2.8}$$

are sufficient to determine the volume fluxes Q_1 and Q_2, as well as y_1, y_2 and x, at a critical point downstream of C_2. The critical point is the point of virtual control defined and discussed by Wood (1968). Although the total volume flux Q is set externally, the flow ratio Q_2/Q_1 is determined by the geometry of the contraction at the critical point.

The equations governing two-layer flow upstream of C_2 are (2.4) together with the energy equation for the upper layer

$$\tfrac{1}{2}\rho_2 Q_2^2/(B^2y_2^2) = (\rho_2-\rho_3)\,g(h_1+h_2-y_1-y_2). \tag{2.9}$$

Equations (2.4) and (2.9) determine $y_1(x)$ and $y_2(x)$ upstream of C_2 once Q_1 and Q_2 are known. The derivatives y_{1x} and y_{2x} satisfy

$$\begin{aligned}\begin{pmatrix}(\rho_1-\rho_2)\,g-\rho_1 Q_1^2/(B^2y_1^3) & \rho_2 Q_2^2/(B^2y_3^2) \\ (\rho_2-\rho_3)\,g & (\rho_2-\rho_3)\,g-\rho_2 Q_2^2/(B^2y_2^3)\end{pmatrix}\begin{pmatrix}y_{1x}\\ y_{2x}\end{pmatrix} \\ = \begin{pmatrix}\rho_1 Q_1^2/(B^3y_1^2)-\rho_2 Q_2^2/(B^3y_2^2) \\ \rho_2 Q_2^2/(B^3y_2^2)\end{pmatrix} B_x.\end{aligned} \tag{2.10}$$

This 2×2 matrix is singular at a critical point upstream of C_2. The condition that the derivatives exist at such a point is that the coefficients of the two rows of (2.10) be proportional. This condition and (2.4), (2.8) and (2.9) are sufficient to determine the volume fluxes Q_1 and Q_2 as well as y_1, y_2 and x at a critical point upstream of C_2.

It can be seen that, in general, there must be only one critical point for which $B_x \neq 0$ in two-layer flow, for otherwise the flow is over-determined. The flow can be subcritical (in terms of the internal Froude number) upstream of C_2 through to a critical point downstream of C_2 in the closed conduit region, or it can have a critical point upstream of C_2 in the open region and be supercritical through and downstream of C_2. Since the equations for a critical point are different on the two sides of C_2, there must be a discontinuity, in general, between the two possible types of flow. The two types of flow are referred to as closed-control flow and open-control flow respectively, and their solutions are now discussed for a contraction of square cross-section.

Solution of (2.4), (2.5) and (2.7) for closed-control flow shows that $y_1 = \frac{4}{5}h_1$ at the critical point, that is, the depth of the lower layer at the critical point is a

† Non-dimensional volume fluxes of the form (2.3) are sometimes referred to as Froude numbers. To be consistent with open-channel flow, it is more correct to define a Froude number here which is less than unity when long waves on the interface can propagate upstream and downstream, and greater than unity when they can only propagate downstream.

constant fraction of its depth in the reservoir. The fraction is the same as that at the contact point C_1 for maximum single-layer flow, because C_1 becomes a critical point as soon as two-layer flow begins. The non-dimensional volume fluxes for two-layer closed-control flow are found to be

$$\frac{\rho_1^{\frac{1}{2}}Q_1}{(\rho_1-\rho_2)^{\frac{1}{2}}g^{\frac{1}{2}}h_1^{\frac{5}{2}}} = \left(\frac{4}{5}\right)^{\frac{3}{2}}\left(\frac{Y_c}{h_1}\right)^{\frac{1}{2}}\left(\frac{Y_c}{h_1}-\frac{2}{5}\right)^{\frac{1}{2}}, \tag{2.11a}$$

$$\frac{\rho_2^{\frac{1}{2}}Q_2}{(\rho_1-\rho_2)^{\frac{1}{2}}g^{\frac{1}{2}}h_1^{\frac{5}{2}}} = \left(\frac{2}{5}\right)^{\frac{1}{2}}\left(\frac{Y_c}{h_1}\right)^{\frac{1}{2}}\left(\frac{Y_c}{h_1}-\frac{4}{5}\right)^{\frac{3}{2}}, \tag{2.11b}$$

where Y_c is the height of the contraction at the critical point. Substitution in (2.8) determines Y_c/h_1, Q_1 and Q_2. The solution is valid provided that the flow is subcritical upstream of C_2. It is necessary also that two-layer contact still occurs at C_2, that is,

$$d(y_1+y_2)/dx \geqslant dY/dx \quad \text{at} \quad C_2.$$

The contact point C_2 is the point at which (2.4), (2.5) and (2.9) are simultaneously valid for the given Q_1 and Q_2.

The only physically valid solution for open-control flow (with the critical point upstream of C_2) is the self-similar solution of Wood (1968), satisfying $y_1/h_1 = y_2/h_2$ at all points upstream of C_2. The flow ratio is independent of the total flux Q, and is found to be

$$\frac{Q_2}{Q_1} = \frac{h_2}{h_1}\left(\frac{(\rho_2-\rho_3)(h_1+h_2)}{(\rho_1-\rho_3)h_1+(\rho_2-\rho_3)h_2}\right)^{\frac{1}{2}}. \tag{2.12}$$

Note that, as the density difference $\rho_1-\rho_2$ between the two layers tends towards zero, $Q_2/Q_1 \rightarrow h_2/h_1$, or as expected, the flow velocities tend towards equality. The solution is valid provided that the flow is supercritical downstream of C_2 and contact is still occurring at C_2. The similarity of the solution enables these conditions to be stated explicitly as

$$\frac{4}{5} < \left[\frac{y_1}{h_1}\right]_{\text{contact}} < \frac{2(\rho_1-\rho_2)h_1h_2+2(\rho_2-\rho_3)(h_1+h_2)^2}{3(\rho_1-\rho_2)h_1h_2+2(\rho_2-\rho_3)(h_1+h_2)^2} \tag{2.13}$$

provided that this range exists. The supercritical condition sets the upper bound and the condition at the contact point sets the lower bound.

Bryant (1974) calculated solutions for the progression from single-layer flow to closed-control two-layer flow, open-control two-layer flow and on to three-layer flow. The method of solution for each type of flow was by analytical reduction of the governing equations as far as possible, followed by the Newton–Raphson numerical method for systems of algebraic equations. In figures 3 (*a*) and (*b*) below, the boundary between single-layer flow and closed-control two-layer flow was found directly from (2.3), and the closed-control solutions were calculated numerically from (2.8) and (2.11 *a*, *b*). The contact point C_2 was calculated numerically from (2.4), (2.5) and (2.9) with Q_1 and Q_2 given by (2.11 *a*, *b*). The upper boundary for closed-control two-layer flow, when

$$dy_1/dx + dy_2/dx = dY/dx$$

at C_2, was then calculated using a refinement of the Newton–Raphson method to avoid difficulties with the zero Jacobian at this maximum flow. Similar approaches using a combination of analytical and numerical methods of solution proved successful for open-control two-layer flow and for three-layer flow.

The calculations showed that, for certain values of the external parameters $\rho_1, \rho_2, \rho_3, h_1, h_2$ and Q, the two types of steady two-layer flow overlapped, while for other values of these parameters, there was a gap between them. In figure 3, an overlap occurs in one part of the two-layer flow region, while there is a gap between the solutions in another part. Where overlap occurs, there are two possible steady two-layer solutions for the given range of external parameters. Where there is a gap between the two types of steady two-layer flow, it was found experimentally that the flow becomes unsteady when the external parameters lie in the gap. The possibility of an overlap has not yet been investigated experimentally, but it seems probable that the flow history would determine which flow occurs. If the overlap is approached by continuous variation of the external parameters from the closed-flow region, it could be expected that the closed flow would persist into the region of overlap, with the reverse occurring if the region of overlap is approached from the open-flow region. In this respect, it would be similar to a supercritical flow incident on a long smooth rise in an open channel, where for the same upstream or downstream controls, dependent only on the flow history, the flow can be supercritical over and beyond the rise or it can pass through a hydraulic jump upstream of the rise and be subcritical over and beyond the rise.

The calculated solutions were continued to three-layer flow. It was found that a smooth transition from two-layer flow to three-layer flow can occur only when the interface at the contact point becomes tangential to the roof of the contraction. This contact point then becomes an additional critical point for the three-layer flow, and moves downstream as the three-layer flow starts. For small density differences between the layers (Boussinesq fluids) the smooth transition to three-layer flow can take place only from open-control two-layer flow, though there is still the possibility of discontinuous transition to three-layer flow from closed-control two-layer flow. For large density differences between the layers, the range of inequality (2.13) does not exist and there is no open-control two-layer flow. A smooth transition to three-layer flow can then take place from closed-control two-layer flow.

The equations governing n-layered flow upstream of the contact point C_n are the equations for the energy differences across the interfaces,

$$\tfrac{1}{2}\rho_k Q_k^2/(B^2y_k^2) - \tfrac{1}{2}\rho_{k+1} Q_{k+1}^2/(B^2y_{k+1}^2) = (\rho_k - \rho_{k+1})\, g(h_k - y_k), \quad k = 1, \ldots, n-1, \tag{2.14}$$

and the energy equation for the upper layer,

$$\tfrac{1}{2}\rho_n Q_n^2/(B^2y_n^2) = (\rho_n - \rho_{n+1})\, g(h_n - y_n). \tag{2.15}$$

The flow downstream of the contact point C_n is governed by the $n-1$ equations (2.14) together with

$$y_1 + y_2 + \ldots + y_n = Y(x). \tag{2.16}$$

The equation determining the layer slopes upstream of C_n is

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & a_{23} & 0 & \cdots & 0 \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ 0 & 0 & 0 & 0 & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} y_{1x} \\ y_{2x} \\ \cdot \\ \cdot \\ \cdot \\ y_{nx} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} B_x, \tag{2.17}$$

where

$$a_{kk} = (\rho_k - \rho_{k+1})\, g - \rho_k Q_k^2/(B^2 y_k^3),$$
$$a_{k,k+1} = \rho_{k+1} Q_{k+1}^2/(B^2 y_{k+1}^3),$$
$$b_k = \rho_k Q_k^2/(B^3 y_k^2) - \rho_{k+1} Q_{k+1}^2/(B^3 y_{k+1}^2).$$

The equation determining the layer slopes downstream of C_n is the same except that the final row of the $n \times n$ matrix is a row of 1's and $b_n = dY/dB$.

The $n \times n$ matrix in (2.17) is singular at no more than the $n-1$ critical points upstream of C_n; at such points a long wave has zero velocity on one of the $n-1$ interfaces. Similarly, the corresponding $n \times n$ matrix for flow downstream of C_n is singular at no more than $n-1$ critical points downstream of C_n. The n-layer flow is determined by a total of exactly $n-1$ critical points in the two parts of the flow, since only then is the total number of unknown variables equal to the total number of available equations. If the n-layer flow resulted from a progression from single-layer flow through two-layer flow and so on, it is difficult to see how there could be less than $n-1$ critical points, because each contact point from C_1 to C_{n-1} becomes a new critical point when one more layer begins to flow smoothly. On the other hand, since there can in principle be up to $n-1$ critical points in both the open and the closed part of the flow, it could easily become over-determined if the total number of critical points in the two parts of the flow exceeded $n-1$. When this occurs the solution fails and the flow becomes unsteady. The n-layer flow in the slow draining of a reservoir, for example, can be expected to consist of intervals during which the flow ratios change smoothly and slowly connected by intervals during which the flow ratios oscillate unsteadily.

3. Experiments

Several experiments were performed to test the major features of the theory and to investigate the unsteady transition between the two types of steady two-layer flow. In the experiments it was desirable for the transition between the two steady states to occur when the depths in the reservoir were reasonably large. It was also necessary to ensure that the unsteady effects should be clearly observable. This suggested that the gap between the two steady solutions should be as wide as possible. The theory showed that these requirements could both be satisfied by withdrawing at a relatively large rate. With the large discharge and the reservoir available (a $4{\cdot}5 \times 1{\cdot}5 \times 1{\cdot}15$ m high glass-sided tank) it would have been difficult to attempt to maintain the level of either layer constant by an inflow mechanism. The reservoir was large enough that for no

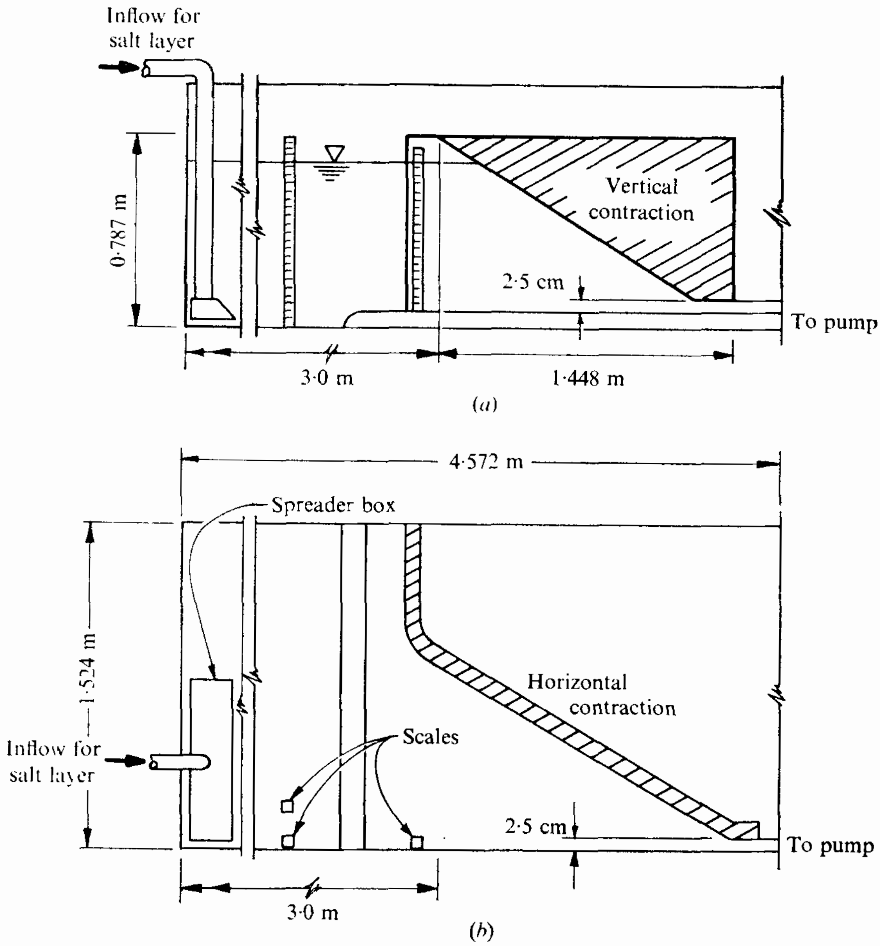

FIGURE 2. Experimental equipment. (*a*) Elevation. (*b*) Plan.

inflow the fall in both layers would be sufficiently slow for the flow to be regarded as steady when it was assumed to be so. It was decided therefore to fill the reservoir with two layers, one fresh water and the other a dyed salt solution, and then to withdraw at a constant discharge.

A timber contraction with a geometry defined by $Y = B = 0{\cdot}6x$ was placed in the reservoir as in figure 2. This contraction had been used for a previous set of experiments, which showed that in spite of the relative steepness of the upper surface the uniform-flow assumption was adequate. An explanation may be found by considering (2.4). This equation relates the velocities on either side of the interface to the height of the interface, and is made usable by the reasonable assumption that the interface velocity may be replaced by the mean velocity $Q/(By)$. For a uniform velocity distribution this assumption is exact, and for more realistic velocity distributions, including an axisymmetric velocity distribution, it is a good approximation. The reservoir was filled with the two layers, and measurements of the levels of each layer were recorded at approximately

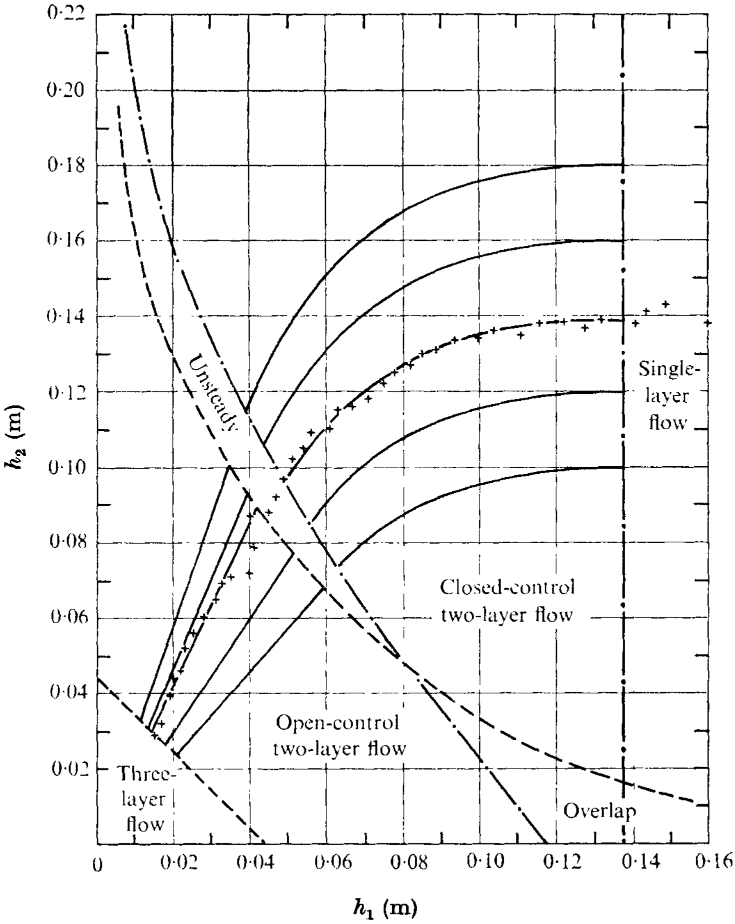

FIGURE 3(*a*). For caption see next page.

2 min intervals after withdrawal began. From the changes in levels, the flow ratio of the two layers was deduced. Samples of the discharge were also taken during each experiment and were analysed later to provide independent verification of the flow ratio of the two layers. Figures 3(*a*) and (*b*) show the results of two experiments. The density difference between the layers in the first experiment was 0·29 % and in the second experiment was 0·78 %. The withdrawal rate in both experiments was $4{\cdot}85 \times 10^{-4}\,\mathrm{m^3/s}$.

The initial depth of the lower layer was chosen to be sufficiently large that withdrawal began with single-layer flow. The first observations are of single-layer flow, and the two-layer flow began at the predicted lower layer depth in both experiments. Because of the scatter in the initial observations, the theoretical solution sketched for the closed-control flow is that which provides the best fit over the initial range of data. Further solutions corresponding to different initial layer depths are sketched for comparison. There is a good fit between the theoretical solution and the experimental data in figure 3(*a*), although the transition between closed-control and open-control flow in the experiment occurs

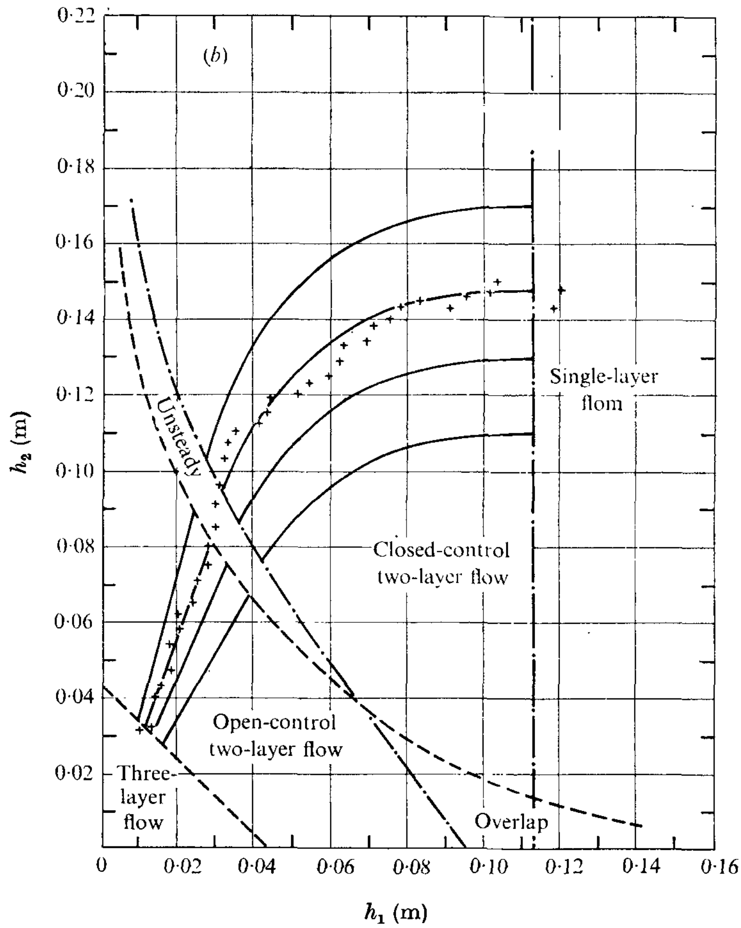

FIGURE 3. (*a*) First experiment: $(\rho_2-\rho_1)/\rho_1 = 0{\cdot}29\,\%$. (*b*) Second experiment: $(\rho_2-\rho_1)/\rho_1 = 0{\cdot}78\,\%$, $Q = 4{\cdot}85\times10^{-4}\text{m}^3/\text{s}$.
+, observations; ——, theory; —·—, boundary of closed-control flow; – – –, boundary of open-control flow.

later than predicted by the theory. The fit between theory and experiment in figure 3 (*b*) is less good, but the transition occurs closer to the position predicted theoretically. It should be remembered that the flow ratio plotted in the figures is determined by the flow at the critical point, and any experimental irregularity at the critical point has a significant effect on the flow ratio. A leaking seal near the critical point in one experiment changed the flow ratio dramatically.

The unsteadiness at the transition consisted of an oscillation of the interface noticeably more rapid than that present during the rest of the experiment and more rapid than could be apparent in observations every 2 min. The theoretical solution for the open-control two-layer flow is a straight line through the origin. The line of best fit is drawn in figures 3 (*a*) and (*b*) together with further lines for comparison. The unsteadiness in the gap between the closed-control and open-control two-layer flows means that there is no simple link between the steady solutions in the two regions. Measurements ceased when the lower layer became too shallow, though it was observed that air entrainment (three-layer flow) began at the total depth predicted.

4. Discussion

The interesting property of steady layered flow is that the flow is controlled by the geometry at the critical points. As critical points move from one part of the flow geometry to another, the flow properties pass through an unsteady transition between two nearly steady regimes. The geometry of the present experiment was simple, but even so, an unsteady transition during the withdrawal of a layered fluid was an essential part of the flow. The flow properties could have been made more complex, for example, by introducing corners in the contraction, because a transition could occur each time a critical point passed over a corner. In general, therefore, the flow of a layered fluid cannot necessarily be expected to change slowly when the external properties change slowly, but instead should be expected to pass through unsteady transitions between intervals in which the variation is slow.

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